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| **1.8** | Limits |
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The concept of *limit* is the underpinning of calculus. In Section [1.7](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-sec1-0007), we said that a function *f* is continuous at *x* = *c* if the values of *f* (*x*) approach *f* (*c*) as *x* approaches *c*. In this section, we define a limit, which makes precise what we mean by approaching.

**The Idea of a Limit**

We first introduce some notation:

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | We write http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math176.gifif the values of *f* (*x*) approach *L* as *x* approaches *c*. | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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How should we find *L*, or even know whether such a number exists? We will look for trends in the values of *f* (*x*) as *x* gets closer to *c*, but *x* ≠ *c*. A graph from a calculator or computer often helps.

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | **Example 1** | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/center_top_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_top_1.gif |

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|  | Use a graph to estimate http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math177.gif. (Use radians.)SolutionFigure [1.81](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0081) shows that as *θ* approaches 0 from either side, the value of sin *θ*/*θ* appears to approach 1, suggesting that http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math178.gif. Zooming in on the graph near *θ* = 0 provides further support for this conclusion. Notice that sin *θ*/*θ* is undefined at *θ* = 0.

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| Figure zoom   | **Figure 1.81** | Find the limit as *θ* → 0 |

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Figure [1.81](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0081) strongly suggests that , but to be sure we need to be more precise about words like “approach” and “close.”

**Definition of Limit**

By the beginning of the 19th century, calculus had proved its worth, and there was no doubt about the correctness of its answers. However, it was not until the work of the French mathematician Augustin Cauchy (1789–1857) that calculus was put on a rigorous footing. Cauchy gave a formal definition of the limit, similar to the following:

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | A function *f* is defined on an interval around *c*, except perhaps at the point *x* = *c*. We define the **limit** of the function *f* (*x*) as *x* approaches *c*, written http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math179.gif, to be a number *L* (if one exists) such that *f* (*x*) is as close to *L* as we want whenever *x* is sufficiently close to *c* (but *x* ≠ *c*). If *L* exists, we write

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Shortly, we see how “as close as we want” and “sufficiently close” are expressed using inequalities. First, we look at more closely (see Example [1](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-mprb-0026)).

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|  | By graphing *y* = (sin *θ*)/*θ* in an appropriate window, find how close *θ* should be to 0 in order to make (sin *θ*)/*θ* within 0.01 of 1.SolutionSince we want (sin *θ*)/*θ* to be within 0.01 of 1, we set the *y*-range on the graphing window to go from 0.99 to 1.01. Our first attempt with -0.5 ≤ *θ* ≤ 0.5 yields the graph in Figure [1.82](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0082). Since we want the *y*-values to stay within the range 0.99 < *y* < 1.01, we do not want the graph to leave the window through the top or bottom. By trial and error, we find that changing the *θ*-range to -0.2 ≤ *θ* ≤ 0.2 gives the graph in Figure [1.83](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0083). Thus, the graph suggests that (sin *θ*)/*θ* is within 0.01 of 1 whenever *θ* is within 0.2 of 0. Proving this requires an analytical argument, not just graphs from a calculator.

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| Figure zoom   | **Figure 1.82** | (sin *θ*)/*θ* with -0.5 ≤ *θ* ≤ 0.5 |

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| Figure zoom   | **Figure 1.83** | (sin *θ*)/*θ* with -0.2 ≤ *θ* ≤ 0.2 |

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When we say “*f* (*x*) is close to *L*,” we measure closeness by the distance between *f* (*x*) and *L*, expressed using absolute values:

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When we say “as close to *L* as we want,” we use ε (the Greek letter epsilon) to specify how close. We write

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to indicate that we want the distance between *f* (*x*) and *L* to be less than ε. In Example [2](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-mprb-0027) we used ε = 0.01. Similarly, we interpret “*x* is sufficiently close to *c*” as specifying a distance between *x* and *c*:

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where δ (the Greek letter delta) tells us how close *x* should be to *c*. In Example [2](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-mprb-0027) we found δ = 0.2.

If , we know that no matter how narrow the horizontal band determined by ε in Figure [1.84](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0084), there is always a δ which makes the graph stay within that band, for *c* - δ < *x* < *c* + δ.

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| Figure zoom   | **Figure 1.84** | What the definition of the limit means graphically |

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Thus we restate the definition of a limit, using symbols:

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | **Definition of Limit**We define http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math185.gifto be the number *L* (if one exists) such that for every ε > 0 (as small as we want), there is a δ > 0 (sufficiently small) such that if |*x* - *c*| < δ and *x* ≠ *c*, then |*f* (*x*) - *L*| < ε. | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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We have arrived at a formal definition of limit. Let's see if it agrees with our intuition.

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|  | Use the definition of limit to show that http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math186.gifSolutionWe must show how, given any ε > 0, we can find a δ > 0 such that

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Since |2*x* - 6| = 2|*x* - 3|, to get |2*x* - 6| < ε we require that |*x* - 3| < ε/2. Thus we take δ = ε/2. |
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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

It is important to understand that the ε, δ definition does not make it easier to calculate limits; rather the ε, δ definition makes it possible to put calculus on a rigorous foundation. From this foundation, we can prove the following properties. See Problems [58](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0403), [59](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0404) and [60](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0405).

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | **Theorem 1.2: Properties of Limits**Assuming all the limits on the right hand side exist:

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| 1.   | If *b* is a constant, then http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math188.gif. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| 2.   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math189.gif. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| 3.   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math190.gif. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| 4.   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math191.gif, provided http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math192.gif. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| 5.   | For any constant *k*, http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math193.gif. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| 6.   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math194.gif. |

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These properties underlie many limit calculations, though we may not acknowledge them explicitly.

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | **Example 4** | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/center_top_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_top_1.gif |

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|  | Explain how the limit properties are used in the following calculation:

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

SolutionWe calculate this limit in stages, using the limit properties to justify each step:

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 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/left_btm_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_btm_1.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

**One- and Two-Sided Limits**

When we write

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| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math197.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

we mean the number that *f* (*x*) approaches as *x* approaches 2 *from both sides*. We examine values of *f* (*x*) as *x* approaches 2 through values greater than 2 (such as 2.1, 2.01, 2.003) and values less than 2 (such as 1.9, 1.99, 1.994). If we want *x* to approach 2 only through values greater than 2, we write

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| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math198.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

for the number that *f* (*x*) approaches (assuming such a number exists). Similarly,

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math199.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

denotes the number (if it exists) obtained by letting *x* approach 2 through values less than 2. We call a *right-hand limit* and a *left-hand limit*. Problems [22](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0367) and [23](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0368) ask for formal definitions of left and right-hand limits.

For the function graphed in Figure [1.85](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0085), we have

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math201.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

If the left- and right-hand limits were equal, that is, if *L*1 = *L*2, then it can be proved that exists and . Since *L*1 ≠ *L*2 in Figure [1.85](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0085), we see that does not exist in this case.

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| Figure zoom   | **Figure 1.85** | Left- and right-hand limits at *x* = 2 |

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 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

**When Limits Do Not Exist**

Whenever there is no number *L* such that , we say does not exist. Here are three examples in which limits fail to exist.

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | **Example 5** | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/center_top_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_top_1.gif |

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|  | Explain why http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math204.gifdoes not exist.SolutionFigure [1.86](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0086) shows the problem: The right-hand limit and the left-hand limit are different. For *x* > 2, we have |*x* - 2| = *x* - 2, so as *x* approaches 2 from the right,

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math205.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

Similarly, if *x* < 2, then |*x* - 2| = 2 - *x* so

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| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math206.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

So if http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math207.gifthen *L* would have to be both 1 and -1. Since *L* cannot have two different values, the limit does not exist.

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| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/image_n/nw0123-y.jpg |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| Figure zoom   | **Figure 1.86** | Graph of |*x* - 2|/(*x* - 2) |

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 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/left_btm_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_btm_1.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | **Example 6** | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/center_top_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_top_1.gif |

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|  | Explain why http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math208.gifdoes not exist.SolutionAs *x* approaches zero, 1/*x*2 becomes arbitrarily large, so it cannot approach any finite number *L*. See Figure [1.87](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0087). Therefore we say 1/*x*2 has no limit as *x* → 0.

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| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/image_n/nw0124-y.jpg |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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| Figure zoom   | **Figure 1.87** | Graph of 1/*x*2 |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/left_btm_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_btm_1.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

If does not exist because *f* (*x*) gets arbitrarily large on both sides of *a*, we also say . So in Example [6](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-mprb-0031) we could say . This behavior may also be described as “diverging to infinity.”

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | **Example 7** | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/center_top_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_top_1.gif |

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|  | Explain why http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math212.gifdoes not exist.SolutionThe sine function has values between -1 and 1. The graph in Figure [1.88](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0088) oscillates more and more rapidly as *x* → 0. There are *x*-values approaching 0 where sin (1/*x*) = -1. There are also *x*-values approaching 0 where sin (1/*x*) = 1. So if the limit existed, it would have to be both -1 and 1. Thus, the limit does not exist.

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| Figure zoom   | **Figure 1.88** | Graph of sin (1/*x*) |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/left_btm_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_btm_1.gif |

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**Limits at Infinity**

Sometimes we want to know what happens to *f* (*x*) as *x* gets large, that is, the end behavior of *f* .

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | If *f* (*x*) gets as close to a number *L* as we please when *x* gets sufficiently large, then we write

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| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math213.gif |  |
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Similarly, if *f* (*x*) approaches *L* when *x* is negative and has a sufficiently large absolute value, then we write

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The symbol ∞ does not represent a number. Writing *x* → ∞ means that we consider arbitrarily large values of *x*. If the limit of *f* (*x*) as *x* → ∞ or *x* → -∞ is *L*, we say that the graph of *f* has *y* = *L* as a *horizontal asymptote*. Problem [24](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0369) asks for a formal definition of lim*x*→∞ *f* (*x*).

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | **Example 8** | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/center_top_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_top_1.gif |

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|  | Investigate http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math215.gifand http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math216.gif.SolutionA graph of *f* (*x*) = 1/*x* in a large window shows 1/*x* approaching zero as *x* increases in either the positive or the negative direction (See Figure [1.89](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0089)). This is as we would expect, since dividing 1 by larger and larger numbers yields answers which are smaller and smaller. This suggests that

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| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math217.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

and that *f* (*x*) = 1/*x* has *y* = 0 as a horizontal asymptote as *x* → ±∞.

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| Figure zoom   | **Figure 1.89** | The end behavior of *f* (*x*) = 1/*x* |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/left_btm_1.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/box/right_btm_1.gif |

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**Definition of Continuity**

We can now give a precise definition of continuity using limits.

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | The function *f* is **continuous** at *x* = *c* if *f* is defined at *x* = *c* and if

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math218.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

In other words, *f* (*x*) is as close as we want to *f* (*c*) provided *x* is close enough to *c*. The function is **continuous on an interval** [*a*, *b*] if it is continuous at every point in the interval.[12](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-note-0012) | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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Constant functions and *f* (*x*) = *x* are continuous for all *x*. Using the continuity of sums and products, we can show that any polynomial is continuous. Proving that sin *x*, cos *x*, and *ex* are continuous is more difficult. The following theorem, based on the properties of limits, makes it easier to decide whether certain combinations of functions are continuous.

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | **Theorem 1.3: Continuity of Sums, Products, and Quotients of Functions**Suppose that *f* and *g* are continuous on an interval and that *b* is a constant. Then, on that same interval,

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| 1.   | *bf*(*x*) is continuous. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| 2.   | *f* (*x*) + *g*(*x*) is continuous. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| 3.   | *f* (*x*)*g*(*x*) is continuous. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| 4.   | *f* (*x*)/*g*(*x*) is continuous, provided *g*(*x*) ≠ 0 on the interval. |

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We prove the third of these properties.

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| Proof | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | Let *c* be any point in the interval. We must show that http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math219.gif. Since *f* (*x*) and *g*(*x*) are continuous, we know that http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math220.gifand http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math221.gif. So, by the first property of limits in Theorem [1.2](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-mthm-0002),

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Since *c* was chosen arbitrarily, we have shown that *f* (*x*)*g*(*x*) is continuous at every point in the interval. |
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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | **Theorem 1.4: Continuity of Composite Functions**If *f* and *g* are continuous, and if the composite function *f* (*g*(*x*)) is defined on an interval, then *f* (*g*(*x*)) is continuous on that interval. | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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Assuming the continuity of sin *x* and *ex*, Theorem [1.4](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-mthm-0004) shows us, for example, that sin (*ex*) and *e*sin *x* are both continuous.

Although we now have a formal definition of continuity, some properties of continuous functions, such as the Intermediate Value Theorem, can be difficult to prove. For a further treatment of limits and continuity, see [www.wiley.com/college/hugheshallett](http://www.wiley.com/college/hugheshallett).

**Exercises and Problems for Section** [**1.8**](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-sec1-0008)

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|  | **Exercises** |
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| **1.** | Use Figure [1.90](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0090) to give approximate values for the following limits (if they exist).

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| (b)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math224.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (c)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math202.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (d)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math225.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/image_n/nw0127-y.jpg |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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| Figure zoom   | **Figure 1.90** |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **2.** | Use Figure [1.91](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0091) to estimate the limits if they exist:

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| (a)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math224.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (b)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math226.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (c)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math202.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (d)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math227.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/image_n/nw0128-y.jpg |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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| Figure zoom   | **Figure 1.91** |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **3.** | Using Figures [1.92](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0092) and [1.93](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-fig-0093), estimate

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| (a)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math228.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (b)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math229.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (c)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math230.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (d)   | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math231.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/image_n/nw0129-y.jpg |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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| Figure zoom   | **Figure 1.92** |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/image_n/nw0130-y.jpg |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
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| Figure zoom   | **Figure 1.93** |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

 | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| Estimate the limits in Exercises [4](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0349) and [5](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0350) graphically. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **4.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math232.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **5.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math233.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **6.** | Does http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math234.gifhave right or left limits at 0? Is *f* (*x*) continuous? |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| Use a graph to estimate each of the limits in Exercises [7](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0352), [8](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0353), [9](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0354) and [10](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0355). Use radians unless degrees are indicated by *θ*°. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **7.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math235.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **8.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math236.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **9.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math237.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **10.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math238.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| For the functions in Exercises [11](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0356), [12](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0357) and [13](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0358), use algebra to evaluate the limits http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math239.gif, and http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math209.gifif they exist. Sketch a graph to confirm your answers. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **11.** | *a* = 4, http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math240.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **12.** | *a* = 2, http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math241.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **13.** | *a* = 3, http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math242.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **14.** | Estimate how close *θ* should be to 0 to make (sin *θ*)/*θ* stay within 0.001 of 1. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **15.** | Write the definition of the following statement both in words and in symbols:

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math243.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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|  | **Problems** |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| In Problems [16](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0361), [17](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0362), [18](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0363) and [19](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0364), is the function continuous for all *x*? If not, say where it is not continuous and explain in what way the definition of continuity is not satisfied. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **16.** | *f* (*x*) = 1/*x* |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **17.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math244.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **18.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math245.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **19.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math246.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **20.** | By graphing *y* = (1 + *x*)1/*x*, estimate http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math247.gif. You should recognize the answer you get. What does the limit appear to be? |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **21.** | What does a calculator suggest about http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math248.gif? Does the limit appear to exist? Explain. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| In Problems [22](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0367), [23](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0368) and [24](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0369), modify the definition of limit to give a definition of each of the following. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **22.** | A right-hand limit |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **23.** | A left-hand limit |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **24.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math249.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **25.** | If *p*(*x*) is the function giving the price of mailing a first-class letter, explain why lim*x*→ 1 *p*(*x*) does not exist. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **26.** | Investigate http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math250.gifnumerically. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **27.** | The notation lim*x*→ 0+ means that we only consider values of *x* greater than 0. Estimate the limit

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math251.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

either by evaluating *xx* for smaller and smaller positive values of *x* (say *x* = 0.1, 0.01, 0.001, …) or by zooming in on the graph of *y* = *xx* near *x* = 0. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| For the functions in Exercises [28](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0373), [29](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0374), [30](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0375), [31](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0376), [32](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0377), [33](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0378), [34](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0379) and [35](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0380), do the following:

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| (a)   | Make a table of values of *f* (*x*) for *x* = 0.1, 0.01, 0.001, 0.0001, -0.1, -0.01, -0.001, and -0.0001. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (b)   | Make a conjecture about the value of http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math224.gif. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (c)   | Graph the function to see if it is consistent with your answers to parts (a) and (b). |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (d)   | Find an interval for *x* near 0 such that the difference between your conjectured limit and the value of the function is less than 0.01. (In other words, find a window of height 0.02 such that the graph exits the sides of the window and not the top or bottom of the window.) |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **28.** | *f* (*x*) = 3*x* + 1 |
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| **29.** | *f* (*x*) = *x*2 - 1 |
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| **30.** | *f* (*x*) = sin 2*x* |
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| **31.** | *f* (*x*) = sin 3*x* |
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| **32.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math252.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **33.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math253.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **34.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math254.gif |
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| **35.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math255.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| Assuming that limits as *x* → ∞ have the properties listed for limits as *x* → *c*, use algebraic manipulations to evaluate http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math256.giffor the functions in Problems [36](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0381), [37](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0382), [38](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0383), [39](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0384), [40](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0385), [41](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0386), [42](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0387), [43](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0388), [44](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0389) and [45](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0390). |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **36.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math257.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **37.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math258.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **38.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math259.gif |
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| **39.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math260.gif |
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| **40.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math261.gif |
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| **41.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math262.gif |
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| **42.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math263.gif |
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| **43.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math264.gif |
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| **44.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math265.gif |
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| **45.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math266.gif |
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| In Problems [46](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0391), [47](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0392), [48](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0393), [49](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0394), [50](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0395), [51](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0396), [52](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0397) and [53](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0398), find a value of the constant *k* such that the limit exists. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **46.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math267.gif |
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| **47.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math268.gif |
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| **48.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math269.gif |
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| **49.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math270.gif |
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| **50.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math271.gif |
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| **51.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math272.gif |
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| **52.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math273.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **53.** | http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math274.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| For each value of ε in Problems [54](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0399) and [55](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0400), find a positive value of δ such that the graph of the function leaves the window *a* - δ < *x* < *a* + δ, *b* - ε < *y* < *b* + ε by the sides and not through the top or bottom. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **54.** | *f* (*x*) = -2*x* + 3; *a* = 0; *b* = 3; ε = 0.2, 0.1, 0.02, 0.01, 0.002, 0.001. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **55.** | *g*(*x*) = -*x*3 + 2; *a* = 0; *b* = 2; ε = 0.1, 0.01, 0.001. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **56.** | Show that http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math275.gif. [Hint: Use Problem [54](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0399).] |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **57.** | Consider the function *f* (*x*) = sin (1/*x*).

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| (a)   | Find a sequence of *x*-values that approach 0 such that sin (1/*x*) = 0.[Hint: Use the fact that sin (π) = sin (2π) = sin (3π) = … = sin (*n*π) = 0.] |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (b)   | Find a sequence of *x*-values that approach 0 such that sin (1/*x*) = 1.[Hint: Use the fact that sin (*n*π/2) = 1 if *n* = 1, 5, 9, ….] |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (c)   | Find a sequence of *x*-values that approach 0 such that sin (1/*x*) = -1. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (d)   | Explain why your answers to any two of parts (a)–(c) show that http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math276.gifdoes not exist. |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **58.** | This problem suggests a proof of the first property of limits: http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math277.gif.

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| (a)   | First, prove the property in the case *b* = 0. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (b)   | Now suppose that *b* ≠ 0. Let ε > 0. Show that if |*f* (*x*) - *L*| < ε/|*b*|, then |*bf*(*x*) - *bL*| < ε. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (c)   | Finally, prove that if http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math176.gifthen http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math278.gif. [Hint: Choose δ so that if |*x* - *c*| < δ, then |*f* (*x*) - *L*| < ε/|*b*|.] |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **59.** | Prove the second property of limits: http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math189.gif. Assume that the limits on the right exist. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **60.** | This problem suggests a proof of the third property of limits (assuming the limits on the right exist):

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math279.gif |  |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

Let http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math280.gifand http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math281.gif.

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| (a)   | First, show that if http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math282.gif, then http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math283.gif. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (b)   | Show algebraically that *f* (*x*)*g*(*x*) = (*f* (*x*) - *L*1) (*g*(*x*) - *L*2) + *L*1*g*(*x*) + *L*2*f* (*x*) - *L*1*L*2. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (c)   | Use the second limit property (see Problem [59](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0404)) to explain why http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math284.gif. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (d)   | Use parts (a) and (c) to explain why http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math285.gif. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |
| (e)   | Finally, use parts (b) and (d) and the first and second limit properties to show that http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/math/math279.gif. |

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| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **61.** | Show *f* (*x*) = *x* is continuous everywhere. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **62.** | Use Problem [61](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-prob-0406) to show that for any positive integer *n*, the function *xn* is continuous everywhere. |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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| **63.** | Use Theorem [1.2](http://edugen.wileyplus.com/edugen/courses/crs3096/rc/hugheshallett9149c01/rc/hugheshallett9149c01/hugheshallett9149c01xlinks.xform?id=hugheshallett9149c01-mthm-0002) to explain why if *f* and *g* are continuous on an interval, then so are *f*  + *g*, *fg*, and *f* /*g* (assuming *g*(*x*) ≠ 0 on the interval). |
| http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif | http://edugen.wileyplus.com/edugen/courses/crs3096/common/art/pixel.gif |

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